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Some results on the Caudrey–Dodd–Gibbon–Kotera–Sawada equation

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Abstract. In this paper, a hierarchy of bilinear Caudrey-Dodd-Gibbon-Kotera-Sawada equations with a unified structure is proposed. A nonlinear superposition formula for the CDGKS equation is proved under certain conditions. A Bäcklund transformation for a higher-order CDGKS equation is presented.

1. Introduction

There are various kinds of generalizations of the celebrated KdV equation to higher-order equations, one of which is the higher-order KdV hierarchy due to Lax [1]. In 1974, Sawada and Kotera gave another higher order KdV equation [2] (also see [3]). Through the dependent variable transformation, we can write this equation as

$$(D_x^6 - D_x D_t) f \cdot f = 0 \tag{1}$$

where the bilinear operator $D_x^m D_t^n$ is defined by [4]

$$D_x^m D_t^n a(x,t) \cdot b(x,t) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t) b(x',t')|_{x'=x, t'=t}.$$

In what follows, we refer to (1) as the Caudrey-Dodd-Gibbon-Kotera-Sawada equation (CDGKS equation). Much research on this equation has been conducted. For example, in [2] the N-soliton solutions of the CDGKS equation were obtained. In 1977, Satsuma and Kaup presented a Bäcklund transformation (BT) for the CDGKS equation in bilinear form [5]

$$(D_x^3 - \lambda) f \cdot f' = 0 \tag{2a}$$

$$(D_t + \tfrac{15}{2}\lambda D_x^2 + \tfrac{3}{2} D_x^5) f \cdot f' = 0 \tag{2b}$$

where λ is a constant parameter (also see [6]). Starting with the BT (2), an infinite number of conserved quantities were also derived. In 1981, Sato and Sato gave a CDGKS hierarchy in bilinear form [7]. Subsequently, Date, Jimbo, Kashiwara and Miwa pointed out that the CDGKS equation can be deduced from the BKP hierarchy under reduction [8, 9]. If we set $x = t_1$, $t = t_5$ and denote $D_{t_1}^6 = D_1^6$, $D_{t_5} D_{t_1} = D_5 D_1$, equation (1) can be written as

$$(D_1^6 - D_1 D_5) f \cdot f = 0 \tag{1'}$$

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Now we put forward the following bilinear CDGKS hierarchy

$$(D_1^6 - D_1 D_5)\tau \cdot \tau = 0$$
$$(9D_1^7 D_m + 5D_m D_7 - 35D_1 D_{m+6} + 21D_1^2 D_5 D_m)\tau \cdot \tau = 0 \tag{3}$$
$$m \text{ is an odd integer, } m \neq 3k,\ m, k \in Z_+.$$

The above equations with $m = 1$, 5, 7, 11 and 13 can be deduced from those obtained in [7]. Note that the CDGKS hierarchy given by (3) possesses a unified structure in the form. As we know, such a simple structure will be easier to treat, and lead to much convenience in calculation when the whole hierarchy of equations are considered. For example, in [10, 11], we have established the nonlinear superposition formulae for the KdV, MKdV and Boussinesq hierarchies respectively with a unified structure. Also in [12] we have obtained the rational solutions of classical Boussinesq hierarchy with a unified structure.

By the use of (A.1)–(A.3), (3) can be rewritten as

$$u_{t_5} = u_{5x} + 5(u_x u_{xx} + u u_{xxx} + u^2 u_x)$$
$$u_{t_7} = u_{7x} + 7(u u_{5x} + 2u_x u_{4x} + 3u_{xx} u_{3x} + 2u^2 u_{3x} + 6u_x u_{xx} u_{3x} + u_x^3 + \tfrac{4}{3}u^3 u_x)$$
$$35 w_{t_{m+6}} = 9u_{5xt_m} + 21 w_{t_m} u_{4x} + 63 u u_{3xt_m} + 105 u_{xt_m} u_{xx} + 35 u^3 w_{t_m} + 105 u^2 u_{xt_m}$$
$$+ 105 w_{t_m} u u_{xx} + 5\partial^{-1} w_{t_m t_7} + 14 w_{t_m} w_{t_5} + 7u\partial^{-1} w_{t_5 t_m} + 21 u_{t_5 t_m} \tag{3'}$$
$$m = 6n+1 \text{ or } 6n+5,\ n \in Z_+ \cup \{0\}.$$

where $u = 6(\ln \tau)_{xx}$, $w_x = u$ and $u_{kx} \equiv \partial^k u/\partial x^k$.

From (3′) we deduce that

$$u_{t_5} = u_{5x} + 5(u_x u_{xx} + u u_{xxx} + u^2 u_x)$$
$$u_{t_{m+6}} = L u_{t_m} \qquad m = 6n+1 \text{ or } 6n+5,\ n \in Z_+ \cup \{0\} \tag{3''}$$

where

$$L = \partial^6 + 6u\partial^4 + 9u_x\partial^3 + (11u_{xx} + 9u^2)\partial^2 + (10u_{3x} + 21uu_x)\partial + (5u_{4x} + 6u_x^2 + 16uu_{xx} + 4u^3)$$
$$+ (u_{5x} + 5uu_{3x} + 5u_x u_{xx} + 5u^2 u_x)\partial^{-1} + u_x \partial^{-1} \cdot (2u_{xx} + u^2).$$

Obviously, (3″) is an equivalent form of the usual CDGKS hierarchy [13, 14] and L is a *recursion operator of the* CDGKS *equation* [14]. In the following discussion, we only focus our attention on the first two equations of CDGKS hierarchy (3). As for the other equations of (3), much work remains to be done.

This paper is organized as follows. In section 2, the CDGKS equation is considered. Under certain conditions, we obtain the corresponding nonlinear superposition formula. A BT for (3) with $m = 1$ is presented in section 3. Finally we list some bilinear operator identities in the appendix which are used in the paper.

2. Nonlinear superposition formula of the CDGKS equation

In this section, under certain conditions, we establish nonlinear superposition formula for the CDGKS equation. In [15], we have considered nonlinear superposition formulae of the Ito equation and a model equation for shallow water waves, which is different from those of the KdV, MKdV equations. We shall see that the nonlinear superposition formula given in the section is the same as that of [15].

In what follows, let f_0 be a solution of the CDGKS equation (1), $f_0 \neq 0$. Suppose that $f_i (i=1,2)$ is a solution of (1) which is related by f_0 under BT(2) with λ_i, i.e. $f_0 \xrightarrow{\lambda_i} f_i$ $(i=1,2)$, and that f_{12} is defined by

$$D_x f_0 \cdot f_{12} = k D_x f_1 \cdot f_2 \qquad \text{(where } k \text{ is a non-zero constant)} \tag{4}$$

From these assumptions, we deduce that

$$\begin{aligned}
0 &= [(D_x^3 - \lambda_1) f_0 \cdot f_1] f_2 - [(D_x^3 - \lambda_2) f_0 \cdot f_2] f_1 \\
&\overset{(A.4)}{=\!=} -3 f_{0xx} D_x f_1 \cdot f_2 + 3 f_{0x} (D_x f_1 \cdot f_2)_x - \frac{1}{4} f_0 [D_x^3 f_1 \cdot f_2 + 3(D_x f_1 \cdot f_2)_{xx}] \\
&\quad + (\lambda_2 - \lambda_1) f_0 f_1 f_2 \\
&\overset{(4)}{=\!=} -\frac{3}{k} f_{0xx} D_x f_0 \cdot f_{12} + \frac{3}{k} f_{0x} (D_x f_0 \cdot f_{12})_x - \frac{1}{4} f_0 \left[D_x^3 f_1 \cdot f_2 + \frac{3}{k} (D_x f_0 \cdot f_{12})_{xx} \right] \\
&\quad + (\lambda_2 - \lambda_1) f_0 f_1 f_2 \\
&= f_0 \left[-\frac{3}{4k} D_x^3 f_0 \cdot f_{12} - \frac{1}{4} D_x^3 f_1 \cdot f_2 + (\lambda_2 - \lambda_1) f_1 f_2 \right]
\end{aligned}$$

which implies that

$$\frac{1}{4} D_x^3 f_1 \cdot f_2 - (\lambda_2 - \lambda_1) f_1 f_2 + \frac{3}{4k} D_x^3 f_0 \cdot f_{12} = 0 \tag{5}$$

and

$$\begin{aligned}
0 &= [(D_x^3 - \lambda_1) f_0 \cdot f_1]_x f_2 - [(D_x^3 - \lambda_2) f_0 \cdot f_2]_x f_1 \\
&\overset{(A.5)}{=\!=} -2 f_{0xxx} D_x f_1 \cdot f_2 + \frac{1}{2} f_{0x} [D_x^3 f_1 \cdot f_2 + 3(D_x f_1 \cdot f_2)_{xx}] \\
&\quad - \frac{1}{2} f_0 [(D_x^3 f_1 \cdot f_2)_x + (D_x f_1 \cdot f_2)_{xxx}] \\
&\quad + f_{0x} (\lambda_2 - \lambda_1) f_1 f_2 - \frac{1}{2} f_0 [(\lambda_1 + \lambda_2) D_x f_1 \cdot f_2 + (\lambda_1 - \lambda_2)(f_1 f_2)_x] \\
&\overset{(4)}{=\!=} -\frac{2}{k} f_{0xxx} D_x f_0 \cdot f_{12} + f_{0x} \left[\frac{1}{2} D_x^3 f_1 \cdot f_2 + (\lambda_2 - \lambda_1) f_1 f_2 + \frac{3}{2k} (D_x f_0 \cdot f_{12})_{xx} \right] \\
&\quad + f_0 \left[-\frac{1}{2} D_x^3 f_1 \cdot f_2 - \frac{1}{2k} (D_x f_0 \cdot f_{12})_{xx} + \frac{1}{2} (\lambda_2 - \lambda_1) f_1 f_2 \right]_x \\
&\quad - \frac{1}{2k} f_0 (\lambda_1 + \lambda_2) D_x f_0 \cdot f_{12} \\
&= f_{0x} \left[\frac{1}{2} D_x^3 f_1 \cdot f_2 - \frac{1}{2k} D_x^3 f_0 \cdot f_{12} + (\lambda_2 - \lambda_1) f_1 f_2 - \frac{1}{k} (\lambda_1 + \lambda_2) f_0 f_{12} \right] \\
&\quad + f_0 \left[-\frac{1}{2} D_x^3 f_1 \cdot f_2 - \frac{1}{2k} D_x^3 f_0 \cdot f_{12} + \frac{1}{2} (\lambda_2 - \lambda_1) f_1 f_2 + \frac{1}{2k} (\lambda_1 + \lambda_2) f_0 f_{12} \right]_x \\
&\overset{(5)}{=\!=} f_{0x} \left[\frac{1}{4k} D_x^3 f_0 \cdot f_{12} + \frac{3}{4} D_x^3 f_1 \cdot f_2 - \frac{1}{k} (\lambda_1 + \lambda_2) f_0 f_{12} \right] \\
&\quad - \frac{1}{2} f_0 \left[\frac{1}{4k} D_x^3 f_0 \cdot f_{12} + \frac{3}{4} D_x^3 f_1 \cdot f_2 - \frac{1}{k} (\lambda_1 + \lambda_2) f_0 f_{12} \right]_x
\end{aligned}$$

which implies that

$$\frac{1}{4k} D_x^3 f_0 \cdot f_{12} + \frac{3}{4} D_x^3 f_1 \cdot f_2 - \frac{1}{k}(\lambda_1+\lambda_2) f_0 f_{12} = c_1(t) f_0^2 \tag{6}$$

where $c_1(t)$ is some function of t. Here and in the following, we assume that there exists a f_{12} determined by (4) such that $c_1(t)=0$, i.e.

$$\frac{1}{4k} D_x^3 f_0 \cdot f_{12} + \frac{3}{4} D_x^3 f_1 \cdot f_2 - \frac{1}{k}(\lambda_1+\lambda_2) f_0 f_{12} = 0 \tag{6'}$$

In this case, we have from (5) and (6′)

$$D_x^3 f_1 \cdot f_2 = \frac{4}{3}\left(\frac{1}{k}(\lambda_1+\lambda_2) f_0 f_{12} - \frac{1}{4k} D_x^3 f_0 \cdot f_{12}\right) \tag{7}$$

$$(\lambda_2-\lambda_1) f_1 f_2 = \frac{1}{3}\left(\frac{2}{k} D_x^3 f_0 \cdot f_{12} + \frac{1}{k}(\lambda_1+\lambda_2) f_0 f_{12}\right) \tag{8}$$

Further, from

$$[(D_t + \tfrac{15}{2}\lambda_1 D_x^2 + \tfrac{3}{2} D_x^5) f_0 \cdot f_1] f_2 - [(D_t + \tfrac{15}{2}\lambda_2 D_x^2 + \tfrac{3}{2} D_x^5) f_0 \cdot f_2] f_1 = 0$$

we can deduce that, by using (A.6), (A.7), (4), (7) and (8),

$$-D_t f_1 \cdot f_2 + \frac{15}{8}(\lambda_1-\lambda_2) D_x^2 f_1 \cdot f_2 - \frac{3}{32} D_x^5 f_1 \cdot f_2 - \frac{45}{8k}(\lambda_1+\lambda_2) D_x^2 f_0 \cdot f_{12} - \frac{45}{32k} D_x^5 f_0 \cdot f_{12} = 0. \tag{9}$$

Similarly, from

$$[(D_x^3 - \lambda_1) f_0 \cdot f_1]_{xx} f_2 - [(D_x^3 - \lambda_2) f_0 \cdot f_2]_{xx} f_1 = 0$$

we can deduce that, by using (4), (7) and (8)

$$-D_x^5 f_1 \cdot f_2 + 4(\lambda_2-\lambda_1) D_x^2 f_1 \cdot f_2 - \frac{4}{k}(\lambda_1+\lambda_2) D_x^2 f_0 \cdot f_{12} + \frac{1}{k} D_x^5 f_0 \cdot f_{12} = 0 \tag{10}$$

Moreover, from

$$\left[\left(D_t + \frac{15}{2}\lambda_1 D_x^2 + \frac{3}{2} D_x^5\right) f_0 \cdot f_1\right]_x f_2 - \left[\left(D_t + \frac{15}{2}\lambda_2 D_x^2 + \frac{3}{2} D_x^5\right) f_0 \cdot f_2\right]_x f_1 + \frac{15}{2}[(D_x^3-\lambda_1) f_0 \cdot f_1]_{xxx} f_2 - \frac{15}{2}[(D_x^3-\lambda_1) f_0 \cdot f_2]_{xxx} f_1 = 0$$

we get, by using (A.5-A.7), (4), (6), (7), (9) and (10),

$$\begin{aligned} f_{0x}&\left[\frac{1}{k} D_t f_0 \cdot f_{12} + \frac{15}{8k}(\lambda_1+\lambda_2) D_x^2 f_0 \cdot f_{12} + \frac{3}{32k} D_x^5 f_0 \cdot f_{12} + \frac{45}{8}(\lambda_2-\lambda_1) D_x^2 f_1 \cdot f_2 + \frac{45}{32} D_x^5 f_1 \cdot f_2\right] \\ &- \frac{1}{2} f_0 \left[\frac{1}{k} D_t f_0 \cdot f_{12} + \frac{15}{8k}(\lambda_1+\lambda_2) D_x^2 f_0 \cdot f_{12} + \frac{3}{32k} D_x^5 f_0 \cdot f_{12} + \frac{45}{8}(\lambda_2-\lambda_1) D_x^2 f_1 \cdot f_2 + \frac{45}{32} D_x^5 f_1 \cdot f_2\right]_x = 0 \end{aligned}$$

which implies that

$$\frac{1}{k}D_tf_0\cdot f_{12}+\frac{15}{8k}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}+\frac{3}{32k}D_x^5f_0\cdot f_{12}+\frac{45}{8}(\lambda_2-\lambda_1)D_x^2f_1\cdot f_2$$
$$+\frac{45}{32}d_x^5f_1\cdot f_2=c_2(t)f_0^2 \tag{11}$$

where $c_2(t)$ is some function of t. Furthermore we assume that f_{12} determined by (4) is chosen such that $c_2(t)=0$, i.e.

$$\frac{1}{k}D_tf_0\cdot f_{12}+\frac{15}{8k}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}+\frac{3}{32k}D_x^5f_0\cdot f_{12}+\frac{45}{8}(\lambda_2-\lambda_1)D_x^2f_1\cdot f_2$$
$$+\frac{45}{32}D_x^5f_1\cdot f_2=0 \tag{12}$$

Then similar to the deduction of (5) and (9), we can get, by using (6′) and (12), that

$$Q_1f_0\equiv[(D_x^3-\lambda_2)f_1\cdot f_{12}]f_0=0$$
$$Q_2f_0\equiv\left[\left(D_t+\frac{15}{2}\lambda_2D_x^2+\frac{3}{2}D_x^5\right)f_1\cdot f_{12}\right]f_0=0$$
$$(D_x^3-\lambda_2)f_1\cdot f_{12}=0$$
$$\left(D_t+\frac{15}{2}\lambda_2D_x^2+\frac{3}{2}D_x^5\right)f_1\cdot f_{12}=0$$

Similarly, we can show that

$$(D_x^3-\lambda_1)f_2\cdot f_{12}=0$$
$$\left(D_t+\frac{15}{2}\lambda_1D_x^2+\frac{3}{2}D_x^5\right)f_2\cdot f_{12}=0.$$

Therefore f_{12} is a new solution of the CDGKS equation (1), which is related by f_1 and f_2.

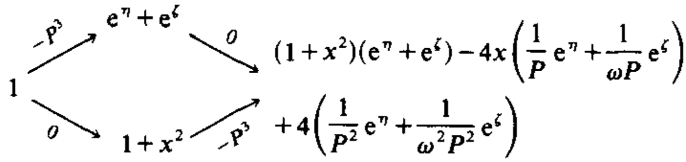
To sum up, we can obtain some particular solutions via the following steps. First choose a given solution f_0 of CDGKS equation (1). Second from the BT (2) we find out f_1 and f_2 such that $f_0\xrightarrow{\lambda_i}f_i$ $(i=1,2)$ and further get a particular solution $\tilde{f}_{12}$ from (4). Then a general solution of (4) is $f_{12}=c(t)f_0+\tilde{f}_{12}$ (where $c(t)$ is an arbitrary function of t). Finally we substitute f_{12} into (6) and (11). If $c(t)$ can be determined such that $c_1(t)=c_2(t)=0$, the corresponding f_{12} is a new solution of the CDGKS equation. For example, we have

$$\begin{array}{ccccc} & \overset{-P_1^3}{\nearrow} & e^{\eta_1}+e^{\zeta_1} & \overset{-P_2^3}{\searrow} & \dfrac{P_1-P_2}{P_1+P_2}e^{\eta_1+\eta_2}+\dfrac{P_1-P_2}{P_1+P_2}e^{\zeta_1+\zeta_2} \\ 1 & & & & \\ & \underset{-P_2^3}{\searrow} & e^{\eta_2}+e^{\zeta_2} & \underset{-P_1^3}{\nearrow} & +\dfrac{\omega P_1-P_2}{\omega P_1+P_2}e^{\eta_2+\zeta_1}+\dfrac{P_1-\omega P_2}{P_1+\omega P_2}e^{\eta_1+\zeta_2} \end{array}$$

where $\eta_i=P_ix-9P_i^5t+\eta_i^0$, $\zeta_i=\omega P_ix-9P_i^5\omega^2t+\zeta_i^0$, P_i, η_i^0 and ζ_i^0 are constants, and $\omega=-1/2+(\sqrt{3}/2)\sqrt{-1}$, $i=1,2$. So

$$\frac{(P_1-P_2)}{(P_1+P_2)}e^{\eta_1+\eta_2}+\frac{(P_1-P_2)}{(P_1+P_2)}e^{\zeta_1+\zeta_2}+\frac{(\omega P_1-P_2)}{(\omega P_1+P_2)}e^{\zeta_1+\eta_2}+\frac{(P_1-\omega P_2)}{(P_1+\omega P_2)}e^{\eta_1+\zeta_2}$$

is a solution of (1)



where $\eta=Px-9P^5t+\eta^0$, $\zeta=\omega Px-9P^5\omega^2t+\zeta^0$, P, η^0 and ζ^0 are constants, and $\omega=-1/2+(\sqrt{3}/2)\sqrt{-1}$. So

$$(1+x^2)(e^{\eta}+e^{\zeta})-4x\left(\frac{1}{P}e^{\eta}+\frac{1}{\omega P}e^{\zeta}\right)+4\left(\frac{1}{P^2}e^{\eta}+\frac{1}{\omega^2P^2}e^{\zeta}\right)$$

is also a solution of (1).

3. A bilinear BT for a higher order CDGKS equation

In this section, we consider equation (3) with $m=1$, i.e.

$$(D_1^6-D_1D_5)\tau\cdot\tau=0 \tag{13a}$$

$$(3D_1^8-10D_1D_7+7D_1^3D_5)\tau\cdot\tau=0 \tag{13b}$$

For (13), we obtain the following results.

Proposition. A BT for (13) is

$$(D_1^3-\lambda)\tau\cdot\tau'=0 \tag{14a}$$

$$(D_5+\tfrac{15}{2}\lambda D_1^2+\tfrac{3}{2}D_1^5)\tau\cdot\tau'=0 \tag{14b}$$

$$(80D_7+840\lambda^2D_1+525\lambda D_1^4+39D_1^7-84D_1^2D_5)\tau\cdot\tau'=0 \tag{14c}$$

where λ is an arbitrary constant.

Proof. Let τ and τ' be two solutions of (13). If we can find three equations which relate τ with τ', and satisfy

$$P_1\equiv\tau'^2(D_1^6-D_1D_5)\tau\cdot\tau-\tau^2(D_1^6-D_1D_5)\tau'\cdot\tau'=0$$

$$P_2\equiv\tau'^2(3D_1^8-10D_1D_7+7D_1^3D_5)\tau\cdot\tau-\tau^2(3D_1^8-10D_1D_7+7D_1^3D_5)\tau'\cdot\tau'=0$$

This is then a BT. Here we show that (14a, b, c) indeed provides a BT for (13).

According to [5], we know that $P_1=0$ can be proved in terms of (14a, b, c). Thus it suffices to show that $P_2=0$. Making use of (A.11)–(A.15), (14a, b, c), P_2 can be rewritten as

$$\begin{aligned}P_2 \xlongequal{(A.11)(A.12)(A.13)} & 3\{\tfrac{7}{4}D_1^5(D_1^3\tau\cdot\tau')\cdot\tau\tau'+\tfrac{7}{2}D_1^3(D_1^5\tau\cdot\tau')\cdot\tau\tau'-\tfrac{13}{4}D_1(D_1^7\tau\cdot\tau')\cdot\tau\tau'\\ &+\tfrac{21}{2}D_1(D_1^5\tau\cdot\tau')\cdot(D_1^2\tau\cdot\tau')+\tfrac{35}{4}D_1(D_1^3\tau\cdot\tau')\cdot(D_1^4\tau\cdot\tau')\\ &+\tfrac{35}{2}D_1^3(D_1^3\tau\cdot\tau')\cdot(D_1^2\tau\cdot\tau')\}-20D_1(D_7\tau\cdot\tau')\cdot\tau\tau'+7\{D_1^3(D_5\tau\cdot\tau')\cdot\tau\tau'\\ &-2D_5(D_1^3\tau\cdot\tau')\cdot\tau\tau'+3D_1[(D_1^2D_5\tau\cdot\tau')\cdot\tau\tau'+(D_5\tau\cdot\tau')\cdot(D_1^2\tau\cdot\tau')]\}\\ =&\tfrac{21}{4}D_1^5(D_1^3\tau\cdot\tau)\cdot\tau\tau'+7D_1^3[(D_5+\tfrac{3}{2}D_1^5)\tau\cdot\tau']\cdot\tau\tau'-\tfrac{39}{4}D_1(D_1^7\tau\cdot\tau')\cdot\tau\tau'\\ &+21D_1[(D_5+\tfrac{3}{2}D_1^5)\tau\cdot\tau']\cdot(D_1^2\tau\cdot\tau')+\tfrac{105}{4}D_1(D_1^3\tau\cdot\tau')\cdot(D_1^4\tau\cdot\tau')\\ &+\tfrac{105}{2}D_1^3(D_1^3\tau\cdot\tau')\cdot(D_1^2\tau\cdot\tau')-20D_1(D_7\tau\cdot\tau')\cdot\tau\tau'\\ &-14D_5(D_1^3\tau\cdot\tau')\cdot\tau\tau'+21D_1(D_1^2D_5\tau\cdot\tau')\cdot\tau\tau'\end{aligned}$$

$$\overset{(14a,b)(\mathrm{A}.14)}{=\!=\!=}-\tfrac{105}{2}\lambda D_1^3(D_1^2\tau\cdot\tau')\cdot\tau\tau'-\tfrac{39}{4}D_1(D_1^7\tau\cdot\tau')\cdot\tau\tau'+\tfrac{105}{4}D_1\lambda\tau\tau'\cdot(D_1^4\tau\cdot\tau')$$
$$+\tfrac{105}{2}D_1^3\lambda\tau\tau'\cdot(D_1^2\tau\cdot\tau')-20D_1(D_7\tau\cdot\tau')\cdot\tau\tau'+21D_1(D_1^2D_5\tau\cdot\tau')\cdot\tau\tau'$$
$$=-105\lambda D_1^3(D_1^2\tau\cdot\tau')\cdot\tau\tau'-\tfrac{39}{4}D_1(D_1^7\tau\cdot\tau')\cdot\tau\tau'-\tfrac{105}{4}\lambda D_1(D_1^4\tau\cdot\tau')\cdot\tau\tau'$$
$$-20D_1(D_7\tau\cdot\tau')\cdot\tau\tau'+21D_1(D_1^2D_5\tau\cdot\tau')\cdot\tau\tau'$$

$$\overset{(\mathrm{A}.15)}{=\!=\!=}-105\lambda D_1[(D_1^4\tau\cdot\tau')\cdot\tau\tau'-2(D_1^3\tau\cdot\tau')\cdot(D_1\tau\cdot\tau')]-\tfrac{39}{4}D_1(D_1^7\tau\cdot\tau')\cdot\tau\tau'$$
$$-\tfrac{105}{4}\lambda D_1(D_1^4\tau\cdot\tau')\cdot\tau\tau'-20D_1(D_7\tau\cdot\tau')\cdot\tau\tau'+21D_1(D_1^2D_5\tau\cdot\tau')\cdot\tau\tau'$$

$$\overset{(14a)}{=\!=\!=}D_1\{(-\tfrac{525}{4}\lambda D_1^4-210\lambda^2D_1-\tfrac{39}{4}D_1^7-20D_7+21D_1^2D_5)\tau\cdot\tau'\}\cdot\tau\tau'$$

$$\overset{(14c)}{=\!=\!=}0.$$

Thus we have completed the proof of the Proposition.

As an application of the BT (14), we can easily obtain the one-soliton solution of (13)

$$\tau=1+\exp(px-9p^5t_5-27p^7t_7+\eta_0)$$

where p, η_0 are constants.

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Appendix

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$(D_x^7D_ta\cdot a)/a^2$$
$$=u_{5xt}+7w_tu_{4x}+21uu_{xxxt}+35u_{xt}u_{xx}+105u^3w_t$$
$$+105u^2u_{xt}+105w_tuu_{xx} \tag{A.1}$$
$$(D_x^2D_yD_ta\cdot a)/a^2=2w_tw_y+u\partial^{-1}w_{ty}+u_{ty} \tag{A.2}$$
$$(D_yD_ta\cdot a)/a^2=\partial^{-1}w_{ty} \tag{A.3}$$

where $u=2(\ln a)_{xx}$, $w_x=u$,

$$(D_x^3a\cdot b)c-(D_x^3a\cdot c)b$$
$$=-3a_{xx}D_xb\cdot c+3a_x(D_xb\cdot c)_x-\tfrac{1}{4}a[D_x^3b\cdot c+3(D_xb\cdot c)_{xx}] \tag{A.4}$$

$$(D_x^3a\cdot b)_xc-(D_x^3a\cdot c)_xb$$
$$=-2a_{xxx}D_xb\cdot c+\tfrac{1}{2}a_x[D_x^3b\cdot c+3(D_xb\cdot c)_{xx}]$$
$$-\tfrac{1}{2}a[(D_x^3b\cdot c)_x+(D_xb\cdot c)_{xxx}] \tag{A.5}$$
$$(D_ta\cdot b)c-(D_ta\cdot c)b=-aD_tb\cdot c \tag{A.6}$$

$$(D_x^5 a\cdot b)c-(D_x^5 a\cdot c)b$$
$$=-5a_{xxxx}D_x b\cdot c+10a_{xxx}(D_x b\cdot c)_x-\tfrac{5}{2}a_{xx}[D_x^3 b\cdot c+3(D_x b\cdot c)_{xx}]$$
$$+\tfrac{5}{2}a_x[(D_x^3 b\cdot c)_x+(D_x b\cdot c)_{xxx}]$$
$$-\tfrac{1}{16}a[D_x^5 b\cdot c+10(D_x^3 b\cdot c)_{xx}+5(D_x b\cdot c)_{xxxx}] \tag{A.7}$$

$$(D_t a\cdot b)_x c-(D_t a\cdot c)_x b=a_t D_x b\cdot c-a_x D_t b\cdot c-\tfrac{1}{2}a[(D_x b\cdot c)_t+(D_t b\cdot c)_x] \tag{A.8}$$

$$(D_x^5 a\cdot b)_x c-(D_x^5 a\cdot c)_x b+5(D_x^3 a\cdot b)_{xxx}c-5(D_x^3 a\cdot c)_{xxx}b$$
$$=-4a_{xxxxx}D_x b\cdot c-10a_{xxxx}(D_x b\cdot c)_x+5a_{xx}[(D_x^3 b\cdot c)_x+(D_x b\cdot c)_{xxx}]$$
$$+\tfrac{1}{4}a_x[D_x^5 b\cdot c+10(D_x^3 b\cdot c)_{xx}+5(D_x b\cdot c)_{xxxx}]$$
$$-\tfrac{3}{8}a[3(D_x^5 b\cdot c)_x+10(D_x^3 b\cdot c)_{xxx}+3(D_x b\cdot c)_{xxxxx}] \tag{A.9}$$

$$\lambda_1(D_x^2 a\cdot b)_x c-\lambda_2(D_x^2 a\cdot c)_x b-\lambda_1(ab)_{xxx}c+\lambda_2(ac)_{xxx}b$$
$$=-2a_x\{(\lambda_1+\lambda_2)(D_x b\cdot c)_x+\tfrac{1}{2}(\lambda_1-\lambda_2)[D_x^2 b\cdot c+(bc)_{xx}]\}$$
$$-2a_{xx}[(\lambda_1+\lambda_2)D_x b\cdot c+(\lambda_1-\lambda_2)(bc)_x] \tag{A.10}$$

$$(D_x^8 a\cdot a)b^2-a^2D_x^8 b\cdot b$$
$$=\tfrac{7}{4}D_x^5(D_x^3 a\cdot b)\cdot ab+\tfrac{7}{2}D_x^3(D_x^5 a\cdot b)\cdot ab-\tfrac{13}{4}D_x(D_x^7 a\cdot b)\cdot ab$$
$$+\tfrac{21}{2}D_x(D_x^5 a\cdot b)\cdot(D_x^2 a\cdot b)-\tfrac{35}{4}D_x(D_x^3 a\cdot b)\cdot(D_x^4 a\cdot b)$$
$$+\tfrac{35}{2}D_x^3(D_x^3 a\cdot b)\cdot(D_x^2 a\cdot b) \tag{A.11}$$

$$(D_x D_t a\cdot a)b^2-a^2D_x D_t b\cdot b=2D_x(D_t a\cdot b)\cdot ab=2D_t(D_x a\cdot b)\cdot ab \tag{A.12}$$

$$(D_x^3 D_y a\cdot a)b^2-a^2D_x^3 D_y b\cdot b$$
$$=D_x^3(D_y a\cdot b)\cdot ab-2D_y(D_x^3 a\cdot b)\cdot ab$$
$$+3D_x[(D_x^2 D_y a\cdot b)\cdot ab+(D_y a\cdot b)\cdot(D_x^2 a\cdot b)] \tag{A.13}$$

$$D_x^{2n+1}a\cdot a=0 \qquad n=0,1,2,\ldots \tag{A.14}$$

$$D_x^3(D_x^2 a\cdot b)\cdot ab=D_x[(D_x^4 a\cdot b)\cdot ab-2(D_x^3 a\cdot b)\cdot(D_x a\cdot b)] \tag{A.15}$$

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